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LIGHTNING LOCATION

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Lightning Location

The Lightning Mapping Array (LMA) is a collection of radio wave Time of Arrival (TOA) sensors. With knowledge of the sensors' positions and TOA's from a source, it is possible to locate the source and find its time of occurrence. The n sensors' locations are specified by (h_j, V_j, λ_j), $j = 1, 2, \ldots, n$, where h denotes altitude above the Earth's surface, V is the geodetic latitude, and λ is the longitude. We will also use spherical coordinates (R,φ, λ), with R being the distance from the center of the Earth and φ denoting spherical latitude. Oblate spheroidal coordinates are given by (ξ, η, λ), where ξ = constant represents an oblate spheroid and η is the spheroidal latitude. The longitudes in all 3 systems are the same. Latitudes are related through

$$\tan V = \frac{a}{b}\tan\eta = \frac{a^2}{b^2}\tan\varphi, \tag{1}$$

where a = 6378137 m and b = 6356752.3142 m are the mean equatorial and polar radii of the Earth respectively [3].

Let $\hat{\mathbf{w}}$ and $\hat{\mathbf{u}}$ be vectors pointing from the center of the Earth to the North Pole and to the intersection of the equator and prime meridian respectively. Define $\hat{\mathbf{v}}$ by $\hat{\mathbf{w}} \times \hat{\mathbf{u}}$. Clearly,

$$u = A\cosh\xi\cos\eta\cos\lambda,\quad v = A\cosh\xi\cos\eta\sin\lambda,\quad w = A\sinh\xi\sin\eta. \tag{2}$$

Here, $A = \sqrt{a^2 - b^2} \approx 521854$ m.

Sensor locations may be written in terms of Cartesian coordinates by using

$$(a\cos\eta + h\cos V)\cos\lambda = u,\quad (a\cos\eta + h\cos V)\sin\lambda = v,\quad b\sin\eta + h\sin V = w, \tag{3}$$

where η is obtained in terms of V by (1). Another transformation is used to place the origin of a new coordinate system at the location of sensor 1. We use

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\sin\lambda_1 & \cos\lambda_1 & 0 \\ -\sin\varphi_1\cos\lambda_1 & -\sin\varphi_1\sin\lambda_1 & \cos\varphi_1 \\ \cos\varphi_1\cos\lambda_1 & \cos\varphi_1\sin\lambda_1 & \sin\varphi_1 \end{pmatrix} \cdot \begin{pmatrix} u - u_1 \\ v - v_1 \\ w - w_1 \end{pmatrix}. \tag{4}$$

In the xyz system, the sensors' z coordinates are not very different compared to other distances in the problem. This may account for more difficulty in retrieving results in this direction, particularly for sources that are distant from the network of sensors. Retrievals of x, y, and t are generally better. A method is suggested to improve the retrieval of the z coordinate of the source.

If we let the position vector $\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$ denote the source location and $\mathbf{r}_j$ be the locations of the sensors, then

$$|\mathbf{r}_j - \mathbf{r}| = c(t_j - t), \tag{5}$$

where c is the speed of light, t is the time of occurrence of the lightning event, and t_j denotes the TOA recorded at sensor j. More sensors are used than are necessary to solve the system of eqs. in (5). The effects of timing errors may be decreased by solving the system in a least squares sense; i.e.,

$$E = \sum_{j=1}^{n} \left[|\mathbf{r}_j - \mathbf{r}| - c(t_j - t) \right]^2 \tag{6}$$

is to be minimized. Apparently, numerical algorithms must be employed to solve the nonlinear problem. Computer time may be reduced by starting with a good estimate of the solution.

Reasonable estimates of the minimum of E might be obtained by working with

$$E = \sum_{j=1}^{n} \left[|\mathbf{r}_j - \mathbf{r}|^2 - c^2 (t_j - t)^2 \right]^2 \tag{7}$$

instead. One method available is to write $|\mathbf{r}_j - \mathbf{r}|^2 = r_j^{\,2} + r^2 - 2\mathbf{r}_j \cdot \mathbf{r}$ and subtract off the nonlinear terms by using $|\mathbf{r}_1 - \mathbf{r}|^2 = c^2 (t_1 - t)^2$ [5]. This leads to a system of linear eqs.

Instead of subtracting the eq. for TOA at sensor 1, we can follow [4] and subtract a TOA eq. averaged over all of the sensors,

$$\frac{1}{n} \sum_{j=1}^{n} |\mathbf{r}_j - \mathbf{r}|^2 = \frac{c^2}{n} \sum_{j=1}^{n} (t_j - t)^2. \tag{8}$$

This leads to the linear system

$$2\left(\mathbf{r}_j - \frac{1}{n} \sum_{k=1}^{n} \mathbf{r}_k \right) \cdot \mathbf{r} - 2c^2 \left(t_j - \frac{1}{n} \sum_{k=1}^{n} t_k \right) t = \left(r_j^{\,2} - \frac{1}{n} \sum_{k=1}^{n} r_k^{\,2} \right) - c^2 \left(t_j^{\,2} - \frac{1}{n} \sum_{k=1}^{n} t_k^{\,2} \right). \tag{9}$$

It was demonstrated in [4] that the averaging approach yielded results almost as good as those obtained by minimizing E in (6).

Even with this method, the retrieval of the z coordinate might be improved. If the x, y, and t results are reasonably good, we can substitute these results back into (7) and minimize E as a function of z alone. This leads to the cubic eq.

$$\frac{\partial E}{\partial z} = -4 \sum_{j=1}^{n} \left[(x_j - x)^2 + (y_j - y)^2 + (z_j - z)^2 - c^2 (t_j - t)^2 \right] (z_j - z) = 0. \tag{10}$$

The solution that yields the smallest value of E is to be used. Define the following sums of sensor coordinates:

$$\frac{1}{n}\sum_{j=1}^{n} x_j^{\,p} = A_p,\ \frac{1}{n}\sum_{j=1}^{n} y_j^{\,p} = B_p,\ \frac{1}{n}\sum_{j=1}^{n} z_j^{\,p} = C_p;\ 1 \le p \le 4;$$
$$\frac{1}{n}\sum_{j=1}^{n} x_j^{\,q} y_j^{\,s} = F_{qs},\ \frac{1}{n}\sum_{j=1}^{n} x_j^{\,q} z_j^{\,s} = G_{qs},\ \frac{1}{n}\sum_{j=1}^{n} y_j^{\,q} z_j^{\,s} = H_{qs};\ 1 \le q,s \le 2. \tag{11}$$

These sums may be evaluated and stored in memory once the sensors are selected. Certain additional sums will have to be performed after arrival times at the sensors are obtained,

$$\frac{1}{n}\sum_{j=1}^{n} \left(ct_j\right)^p = D_p,\ 1 \le p \le 4;$$
$$\frac{1}{n}\sum_{j=1}^{n} x_j^{\,q}\left(ct_j\right)^s = J_{qs},\ \frac{1}{n}\sum_{j=1}^{n} y_j^{\,q}\left(ct_j\right)^s = K_{qs},\ \frac{1}{n}\sum_{j=1}^{n} z_j^{\,q}\left(ct_j\right)^s = L_{qs};\ 1 \le q,s \le 2. \tag{12}$$

In terms of the sums defined above, the sum of the squared residuals is given by

$$\begin{aligned}
E &= z^4 \\
&\quad - \{4C_1\}z^3 \\
&\quad + \{2[(A_2 + B_2 + 3C_2 - D_2) - 2(A_1 x + B_1 y - D_1 ct) + (x^2 + y^2 - c^2t^2)]\}z^2 \\
&\quad - \{4[(G_{21} + H_{21} - L_{12} + C_3) - 2(G_{11} x + H_{11} y - L_{11} ct) + C_1(x^2 + y^2 - c^2t^2)]\}z \\
&\quad + \{2(F_{22} + G_{22} + H_{22} - J_{22} - K_{22} - L_{22}) + A_4 + B_4 + C_4 + D_4 \\
&\qquad - 4[(A_3 + F_{12} + G_{12} - J_{12})x + (B_3 + F_{21} + H_{12} - K_{12})y + (D_3 - J_{21} - K_{21} - L_{21})ct] \\
&\qquad + 2[(3A_2 + B_2 + C_2 - D_2)x^2 + (A_2 + 3B_2 + C_2 - D_2)y^2 - (A_2 + B_2 + C_2 - 3D_2)c^2t^2] \\
&\qquad + 8(F_{11} xy - J_{11} xct - K_{11} yct) - 4(A_1 x^3 + B_1 y^3 + D_1 c^3t^3) \\
&\qquad - 4(B_1 x^2 y + A_1 xy^2 - D_1 x^2 ct - A_1 xc^2t^2 - D_1 y^2 ct - B_1 yc^2t^2) \\
&\qquad + x^4 + y^4 + c^4t^4 + 2x^2y^2 - 2(x^2 + y^2)c^2t^2\}
\end{aligned} \tag{13}$$

The cubic eq. obtained by setting $\partial E/\partial z = 0$,

$$\begin{gathered}
z^3 + a_2 z^2 + a_1 z + a_0 = 0; \\
a_2 = -3C_1, \\
a_1 = A_2 + B_2 + 3C_2 - D_2 - 2(A_1 x + B_1 y - D_1 ct) + x^2 + y^2 - c^2t^2, \\
a_0 = -[(G_{21} + H_{21} - L_{12} + C_3) - 2(G_{11} x + H_{11} y - L_{11} ct) + C_1(x^2 + y^2 - c^2t^2)],
\end{gathered} \tag{14}$$

may be solved exactly [1]. The coordinates as given by the procedure described may be used as a starting point for a numerical method (e.g., Newton, Marquardt [2]) to minimize E.

Once x, y, z, and t are obtained to sufficient accuracy, the inverse of (4) may be employed to obtain the source location in terms of the Cartesian coordinates u, v, and w. From these, longitude is immediately obtained from $\tan\lambda = v/u$. For the latitude and altitude, we begin with

$$\rho\tilde{\mathbf{n}} + w\hat{\mathbf{w}} = \rho_0\tilde{\mathbf{n}} + w_0\hat{\mathbf{w}} + \mu\mathbf{N}, \quad \rho^2 = u^2 + v^2, \quad \mathbf{N} = \frac{\rho_0}{a^2}\tilde{\mathbf{n}} + \frac{w_0}{b^2}\hat{\mathbf{w}},$$
$$\rho_0 = \rho/\left(1 + \mu/a^2\right), \quad w_0 = w/\left(1 + \mu/b^2\right), \tag{15}$$

where ρ_0 and w_0 correspond to a point on the surface of the Earth directly below the source. Substituting into the eq. of the oblate spheroid taken as the Earth's surface yields

$$\mu^4 + 2\left(a^2 + b^2\right)\mu^3 + \left[\left(a^2 + b^2\right)^2 + 2a^2b^2 - a^2\rho^2 - b^2w^2\right]\mu^2$$
$$+ 2a^2b^2\left(a^2 + b^2 - \rho^2 - w^2\right)\mu + a^2b^2\left(a^2b^2 - b^2\rho^2 - a^2w^2\right) = 0. \tag{16}$$

Once the unknown constant μ is obtained, the associated coordinates ρ_0 and w_0 may be recovered by (15). The geodetic latitude and altitude may be calculated from

$$V = \tan^{-1}\frac{a^2w_0}{b^2\rho_0}, \quad h = \sqrt{\left(\rho - \rho_0\right)^2 + \left(w - w_0\right)^2}. \tag{17}$$

Since spherical and geodetic latitudes are not very different, we may simply perturb;

$$\eta \approx \tan^{-1}\left(\frac{aw}{b\rho}\right) - \frac{\left(a^2 - b^2\right)\rho w\left(\sqrt{a^2w^2 + b^2\rho^2} - ab\right)}{\left(a^2 - b^2\right)\left(a^2w^2 - b^2\rho^2\right) + ab\left(\rho^2 + w^2\right)\sqrt{a^2w^2 + b^2\rho^2}}. \tag{18}$$

The geodetic latitude may then be recovered by (1), and the altitude may be determined from (3).

References

1. Abranowitz, M. and I. Stegun (1970), *Handbook of Mathematical Functions*, 9th Printing, Dover, NY, 17-18.
2. Burden, R. and J. Faires (1993), *Numerical Analysis*, 5th Ed., PWS-Kent, Boston, 553-576.
3. Defense Mapping Agency (DMA) (1991), Department of Defense World Geodetic System 1984, 2nd Ed., Technical Report TR83502, 3-1, 3-10.
4. Hager, W. and D. Wang (1995), "An analysis of errors in the location, current, and velocity of lightning", *J. Geophys. Res.*, 100, 25,721-25,729.
5. Koshak, W. and R. Solakiewicz (2001), "TOA lightning location retrieval on spherical and oblate spheroidal earth geometries", *J. Atmos. Oceanic Technol.*, 18, 187-199.